Additional Mathematics Notes

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1 Quadratic Equations & Inequalities

Sum & Product Of Roots

Sum of roots $= -\dfrac{b}{a}$

Product of roots $= \dfrac{c}{a}$

Quadratic Equation From Roots

$x^2 -$ (sum of roots) $x +$ (product of roots) $= 0$

2, 1 or 0 real roots

2 real roots: $b^2 - 4ac > 0$

1 real root (2 equal roots): $b^2 - 4ac = 0$

0 real roots: $b^2 - 4ac < 0$

Curve Always Positive / Negative

$b^2 - 4ac < 0$ (because curve has 0 real roots)

Line & Curve

Line intersect curve (at 2 points): $b^2 - 4ac > 0$

Line tangent to curve: $b^2 - 4ac = 0$

Line does not intersect curve: $b^2 - 4ac < 0$

*Line meets curve: $b^2 - 4ac \geq 0$

2 Indices & Surds

Indices

1. $a^m \times a^n = a^{m+n}$
2. $a^m \div a^n = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $a^0 = 1$ where $a \neq 0$
5. $a^{-n} = \frac{1}{a^n}$
6. $a^{\frac{1}{n}} = \sqrt[n]{a}$
7. $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$
8. $(a \times b)^n = a^n \times b^n$
9. $(\frac{a}{b})^n = \frac{a^n}{b^n}$

Surds

1. $\sqrt{a} \times \sqrt{a} = a$
2. $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
3. $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
4. $m\sqrt{a} + n\sqrt{a} = (m+n)\sqrt{a}$
5. $m\sqrt{a} - n\sqrt{a} = (m-n)\sqrt{a}$

Rationalise Denominator

For $\dfrac{k}{a\sqrt{b}}$, multiply numerator and denominator by $\sqrt{b}$.

For $\dfrac{k}{a\sqrt{b} + c\sqrt{d}}$, multiply by the conjugate, which is $a\sqrt{b} - c\sqrt{d}$.

3 Polynomials & Partial Fractions

Polynomial Division

$P(x) = \text{divisor} \times Q(x) + R(x)$

Remainder Theorem

If $P(x)$ is divided by $x - c$, remainder is $\mathrm{f}(c)$.

If $P(x)$ is divided by $ax - b$, remainder is $\mathrm{f}\left(\dfrac{b}{a}\right)$.

Factor Theorem

If $x + c$ is a factor of $P(x)$, $\mathrm{f}(-c) = 0$.

If $ax + b$ is a factor of $P(x)$, $\mathrm{f}\left(-\dfrac{b}{a}\right) = 0$.

Cubic Polynomials

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Partial Fractions

1. $\dfrac{f(x)}{(ax+b)(cx+d)} = \dfrac{A}{ax+b} + \dfrac{B}{cx+d}$
2. $\dfrac{f(x)}{(ax+b)(cx+d)^2} = \dfrac{A}{ax+b} + \dfrac{B}{cx+d} + \dfrac{C}{(cx+d)^2}$
3. $\dfrac{f(x)}{(ax+b)(x^2+c)} = \dfrac{A}{ax+b} + \dfrac{Bx+C}{x^2+c}$

Special case: $\dfrac{f(x)}{(ax+b)(x^2)} = \dfrac{A}{ax+b} + \dfrac{B}{x} + \dfrac{C}{x^2}$

4 Binomial Expansions

Binomial Expansions

$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + ... + \binom{n}{r}a^{n-r}b^r + ... + b^n$

General Term

$T_{r+1} = \binom{n}{r}a^{n-r}b^r$

n choose r

$\binom{n}{r} = \dfrac{n!}{r!(n-r)!} = \dfrac{n(n-1)...(n-r+1)}{r!}$

5 Power, Exponential, Logarithmic & Modulus Functions

Modulus Functions

For $|a| = b \Rightarrow a = b$ or $a = -b$.

Logarithm Definition

For $\log_a y$ to be defined,

1. $y > 0$
2. $a > 0, a \neq 1$

Laws Of Logarithms

1. $\log_a x^n = n \log_a x$
2. $\log_a xy = \log_a x + \log_a y$
3. $\log_a \frac{x}{y} = \log_a x - \log_a y$
4. $\log_a b = \dfrac{\log_c b}{\log_c a}$
5. $\log_a b = \dfrac{1}{\log_b a}$

Logarithms To Exponential

$\log_a y = x \Leftrightarrow y = a^x$

$\lg y = x \Leftrightarrow y = 10^x$

$\ln y = x \Leftrightarrow y = e^x$

6 Trigonometric Functions, Identities & Equations

Special Angles

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	$\frac{\sqrt{0}}{2} = 0$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2} = 1$
$\cos\theta$	$\frac{\sqrt{4}}{2} = 1$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{0}}{2} = 0$
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	–

Reciprocal Functions

1. $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$
2. $\sec\theta = \dfrac{1}{\cos\theta}$
3. $\cot\theta = \dfrac{1}{\tan\theta}$

Negative Functions

1. $\sin(-\theta) = -\sin\theta$
2. $\cos(-\theta) = \cos\theta$
3. $\tan(-\theta) = -\tan\theta$

Tangent & Cotangent

1. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
2. $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

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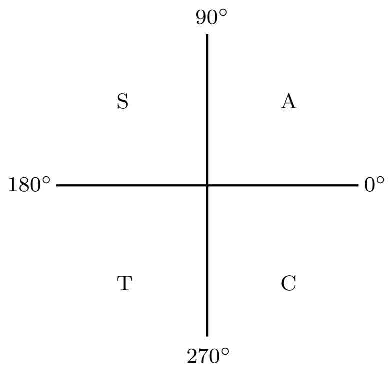


Trigonometric Identities

1. $\sin^2 A + \cos^2 A = 1$
2. $\sec^2 A = 1 + \tan^2 A$
3. $\mathrm{cosec}^2 A = 1 + \cot^2 A$

Addition Formulae

1. $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
2. $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
3. $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

Double Angle Formulae

1. $\sin 2A = 2 \sin A \cos A$
2. $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$
3. $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

R-Formulae

For $a > 0, b > 0, 0° < \alpha < 90°$,
1. $a \sin\theta \pm b\cos\theta = R\sin(\theta \pm \alpha)$
2. $a \cos\theta \pm b\sin\theta = R\cos(\theta \mp \alpha)$
where $R = \sqrt{a^2 + b^2}, \tan\alpha = \dfrac{b}{a}$.

Principal Values

1. $-\dfrac{\pi}{2} \leq \sin^{-1}\theta \leq \dfrac{\pi}{2}$
2. $0 \leq \cos^{-1}\theta \leq \pi$
3. $-\dfrac{\pi}{2} < \tan^{-1}\theta < \dfrac{\pi}{2}$

7 Transformation Of Trigonometric Graphs

Transformation to $y = a\sin x / a\cos x / a\tan x$

1. If $a > 0$: Scaling of graph with a factor of a parallel to the y-axis 2. If $a < 0$: Scaling of graph with a factor of a parallel to the y-axis, then reflecting of graph in x-axis

For sin & cos: amplitude becomes $|a|$
For tan: there is no amplitude

amplitude $= \dfrac{\text{maximum} - \text{minimum}}{2}$

Transformation to $y = \sin bx / \cos bx / \tan bx$

Scaling of graph with a factor of $\dfrac{1}{b}$ parallel to the x-axis

For sin & cos: period becomes $\dfrac{2\pi}{b}$

For tan: period becomes $\dfrac{\pi}{b}$

Transformation to $y = \sin x + c / \cos x + c / \tan x + c$

Translating of graph by c units parallel to the y-axis

$c = \dfrac{\text{maximum} + \text{minimum}}{2}$

Transformation to $y = a\sin bx + c$

1. $y = \sin bx$: Scaling of graph with a factor of $\dfrac{1}{b}$ parallel to the x-axis 2. $y = a\sin bx$: Scaling of graph with a factor of a parallel to the y-axis (reflecting of graph in x-axis if $a < 0$) 3. $y = a\sin bx + c$: Translating of graph by c units parallel to the y-axis

8 Coordinate Geometry

Gradient

$m = \dfrac{y_1 - y_2}{x_1 - x_2}$

Equation

$y - y_1 = m(x - x_1)$
$y = mx + c$

Midpoint

$\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Parallel Lines

$m_1 = m_2$

Perpendicular Lines

$m_1 = -\dfrac{1}{m_2}$
$m_1 \times m_2 = -1$

Area Of Quadrilateral

$A = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{vmatrix}$
$= \frac{1}{2}|(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (x_2y_1 + x_3y_2 + x_4y_3 + x_1y_4)|$

Note: coordinates should be in anti-clockwise direction

Circle

$(x - a)^2 + (y - b)^2 = r^2$
(a, b): centre of circle
r: radius

$x^2 + y^2 + 2gx + 2fy + c = 0$
$(-g, -f)$: centre of circle
$\sqrt{f^2 + g^2 - c}$: radius

9 Differentiation

Differentiation Rules

1. $\frac{d}{dx}c = 0$
2. $\frac{d}{dx}x^n = nx^{n-1}$
3. $\frac{d}{dx}\sin x = \cos x$
4. $\frac{d}{dx}\cos x = -\sin x$
5. $\frac{d}{dx}\tan x = \sec^2 x$
6. $\frac{d}{dx}e^x = e^x$
7. $\frac{d}{dx}\ln\ x = \frac{1}{x}$

Note: $\frac{d}{dx}k\mathrm{f}(x) = k \times \frac{d}{dx}\mathrm{f}(x)$

Chain Rule

For $y = \mathrm{f}(u)$ and $u = \mathrm{g}(x)$,
$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

Further Differentiation Rules (Chain Rule)

1. $\frac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$
2. $\frac{d}{dx}\sin(ax + b) = a\cos(ax + b)$
3. $\frac{d}{dx}\cos(ax + b) = -a\sin(ax + b)$
4. $\frac{d}{dx}\tan(ax + b) = a\sec^2(ax + b)$
5. $\frac{d}{dx}e^{ax+b} = ae^{ax+b}$
6. $\frac{d}{dx}\ln\ (ax + b) = \frac{a}{ax+b}$

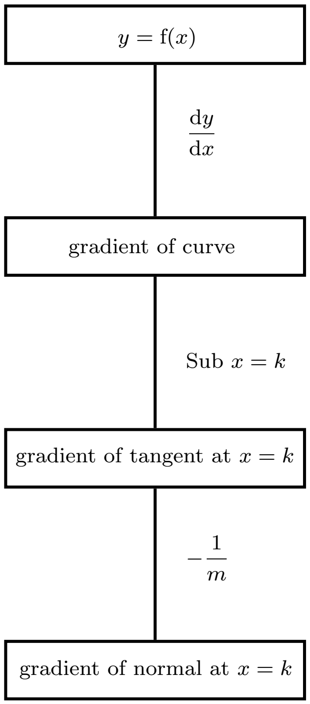
Product Rule

$\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

Quotient Rule

$\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Gradient Of Curve, Tangent & Normal



Increasing & Decreasing Functions

1. For increasing functions, $\frac{dy}{dx} > 0$.
2. For decreasing functions, $\frac{dy}{dx} < 0$.

Rates Of Change

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

Stationary point

A stationary point is defined when $\frac{dy}{dx} = 0$.

First derivative test

If $\frac{dy}{dx} = 0$ for $x = k$, test for k^-, k, k^+.

Maximum point:

x	k^-	k	k^+
$\frac{dy}{dx}$	$+$	0	$-$

Minimum point:

x	k^-	k	k^+
$\frac{dy}{dx}$	$-$	0	$+$

Inflexion point:

x	k^-	k	k^+
$\frac{dy}{dx}$	$+$	0	$+$
$\frac{dy}{dx}$	$-$	0	$-$

Second Derivative Test

1. If $\frac{d^2y}{dx^2} < 0$, it is a maximum point.
2. If $\frac{d^2y}{dx^2} > 0$, it is a minimum point.
3. If $\frac{d^2y}{dx^2} = 0$, need to do first derivative test.

10 Integration

Integration Rules

1. $\int k \, dx = kx + c$
2. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$
3. $\int \sin x \, dx = -\cos x + c$
4. $\int \cos x \, dx = \sin x + c$
5. $\int \sec^2 x \, dx = \tan x + c$
6. $\int e^x \, dx = e^x + c$
7. $\int \frac{1}{x} \, dx = \ln x + c$

Note: $\int k f(x) \, dx = k \times \int f(x) \, dx$

Further Integration Rules

1. $\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c, n \neq -1$
2. $\int \sin(ax+b) \, dx = -\frac{\cos(ax+b)}{a} + c$
3. $\int \cos(ax+b) \, dx = \frac{\sin(ax+b)}{a} + c$
4. $\int \sec^2(ax+b) \, dx = \frac{\tan(ax+b)}{a} + c$
5. $\int e^{ax+b} \, dx = \frac{e^{ax+b}}{a} + c$
6. $\int \frac{1}{ax+b} \, dx = \frac{\ln(ax+b)}{a} + c$

Definite Integral

For $\int f(x) \, dx = F(x) + c$,
$\int_a^b f(x) \, dx = F(b) - F(a)$.

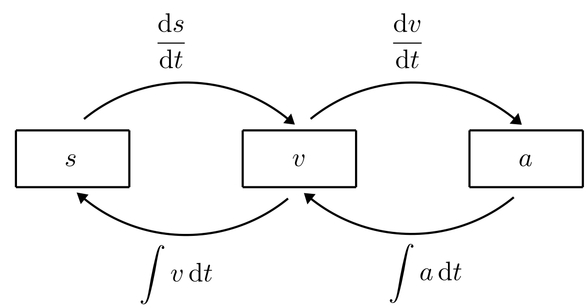
Area With Respect To x-axis Or y-axis

For area with respect to x-axis, $\int_a^b f(x) \, dx$.

For area with respect to y-axis, $\int_c^d f(y) \, dy$.

Note: For area below the x-axis (taken with respect to x-axis) or area to the left of the y-axis (taken with respect to y-axis), it is taken as negative.

Kinematics



1. $v = \frac{ds}{dt}$
2. $a = \frac{dv}{dt}$
3. $s = \int v \, dt$
4. $v = \int a \, dt$

Note:

a. velocity, v determines both the *speed* and the *direction*
b. average speed $= \frac{\text{total distance}}{\text{total time}}$
c. particle starts from origin, $s = 0$
d. instantaneously at rest, $v = 0$
e. max / min velocity, $a = \frac{dv}{dt} = 0$
f. max / min displacement, $v = \frac{ds}{dt} = 0$